

Chaotic and Predictable Representations for Multidimensional Lévy Processes[☆]

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Abstract

For a general Multidimensional Lévy process (satisfying some moment conditions), we introduce the Multidimensional power jump processes and the related Multidimensional Teugels martingales. Furthermore, we orthogonalize the Multidimensional Teugels martingales by applying Gram-Schmidt process. We give a chaotic representation for every square integral random variable in terms of these orthogonalized Multidimensional Teugels martingales. The predictable representation with respect to the same set of Multidimensional orthogonalized martingales of square integrable random variables and of square integrable martingales is an easy consequence of the chaotic representation.

Keywords: Lévy processes, Martingales, Stochastic integration, Orthogonal polynomials.

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1. Introduction

The chaotic representation property (CRP) has been studied by Emery (1989) for normal martingales, that is, for martingales X such that $\langle X, X \rangle_t = ct$, for some constant $c > 0$. This property says that any square integrable random variable measurable with respect to X can be expressed as an orthogonal sum of multiple stochastic integrals with respect to X . It is known (see for example Dellacherie et al., 1992, p. 207 and Dermoune, 1990), that the only normal martingales X , with the CRP, or even the weaker predictable representation property (PRP), which are also Lévy processes are the Brownian motion and the compensated Poisson process. David Nualart and Wim Schoutens (2000) study the chaotic representation property for one-dimensional Lévy process, in terms of a suitable orthogonal sequence of martingales where these martingales are obtained as the orthogonalization of the compensated power jump processes of the Lévy process. Furthermore, Nualart and Schoutens (2001) used their martingale representation result to establish the existence and uniqueness of solutions for BSDE's driven by a Lévy process of the kind considered in Nualart and Schoutens (2000). In addition, Corcuera, Nualart and Schoutens (2005) applied this martingale representation result to study the completion of a Lévy market by power-jump assets.

In the past twenty years, there is already a growing interest for multidimensional Lévy Processes. Some concepts and basic properties about multidimensional Lévy Processes were summarized in Sato (1999). Applications of multidimensional Lévy Processes to analyzing biomolecular (DNA and protein) data and one-server light traffic queues were explored by Dembo, Karlin and Zeittouni (1994). A small deviations property of multidimensional Lévy Processes were discussed by Simon (2003). In finance research, practically all financial applications require a multivariate model with dependence between components: examples are basket option pricing, portfolio optimization, simulation of risk scenarios for portfolios. In most of these applications, jumps in the price process must be taken into account. Cont and Tankov (2004) systematically investigated these problems in multidimensional Lévy market. In addition, the optimal portfolios in multidimensional Lévy market is discussed by Emmer and Klüppelberg (2004). Some simulation approaches for multivariate Lévy processes are also investigated in Cohen and Rosiński (2007). Lévy copulas was also suggested by Kallsen and Tankov (2006) in order to characterize the dependence among components

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of multidimensional Lévy Processes. The SDEs driven by infinite-dimensional Lévy processes was investigated by Meyer-Brandis and Proske (2010).

The chaotic representation property is important for the research of Lévy process, and multidimensional Lévy processes obtain some applications in bioscience and finance, so it is significant to extend the result in univariate set-up obtained by Nualart and Schouten(2000) to the cases of multidimensional Lévy processes. In this paper, following the research line of the paper in Nualart and Schouten(2000), we study the chaotic representation property for Multidimensional Lévy processes, in terms of a suitable orthogonal sequence of Multidimensional martingales, assuming that the Lévy measure has a finite Laplace transform outside the origin. These Multidimensional martingales are obtained as the orthogonalization of the Multidimensional compensated power jump processes of our Multidimensional Lévy process. In Section 2, we introduce these Multidimensional compensated power jump processes and we transform them into a multivariate orthogonal sequence. Section 3 is devoted to prove the chaos representation property from which a predictable representation is deduced. Finally, in Section 4, we discuss some particular examples.

2. Preliminary

A \mathbb{R}^n -valued stochastic process $X = \{X(t) = (X_1(t), X_2(t), \dots, X_n(t))', t \geq 0\}$ defined in complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *Lévy process* if X has stationary and independent increments and $X(0) = \mathbf{0}$. A Lévy process possesses a càdlàg modification (Protter,1990, Theorem 30,p.21) and we will always assume that we are using this càdlàg version. If we let $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}$, where $\mathcal{G}_t = \sigma\{X(s), 0 \leq s \leq t\}$ is the natural filtration of X , and \mathcal{N} are the \mathbb{P} -null sets of \mathcal{F} , then $\{\mathcal{F}_t, t \geq 0\}$ is a right continuous family of σ -fields (Protter,1990,Theorem 31,p.22). We assume that \mathcal{F} is generated by X . For an up-to-date and comprehensive account of Lévy processes we refer the reader to Bertoin (1996) and Sato (1999).

Let X be a Lévy process and denote by

$$X(t-) = \lim_{s \rightarrow t-, s < t} X(s), \quad t > 0,$$

the left limit process and by $\Delta X(t) = X(t) - X(t-)$ the jump size at time t . It is known that the law of $X(t)$ is *infinitely divisible* with characteristic function of the form

$$E[\exp(i\theta \cdot X(t))] = (\phi(\theta))^t, \quad \theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n$$

where $\phi(\theta)$ is the characteristic function of $X(1)$. The function $\psi(\theta) = \log \phi(\theta)$ is called the *characteristic exponent* and it satisfies the following famous Lévy-Khintchine formula (Bertoin, 1996):

$$\psi(\theta) = -\frac{1}{2}\theta \cdot \Sigma \theta + i\mathbf{a} \cdot \theta + \int_{\mathbb{R}^n} (\exp(i\theta \cdot \mathbf{x}) - 1 - i\theta \cdot \mathbf{x}1_{|\mathbf{x}| \leq 1}) \nu(d\mathbf{x}).$$

where $\mathbf{a}, \mathbf{x} \in \mathbb{R}^n$, Σ is a symmetric nonnegative-definite $n \times n$ matrix, and ν is a measure on $\mathbb{R}^n \setminus \{o\}$ with $\int (\|\mathbf{x}\|^2 \wedge 1) \nu(d\mathbf{x}) < \infty$. The measure ν is called the *Lévy measure* of X .

Throughout this paper, we will use the standard multi-index notation. We denote by \mathbb{N}_0 the set of nonnegative integers. A multi-index is usually denoted by \mathbf{p} , $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathbb{N}_0^n$. Whenever \mathbf{p} appears with subscript or superscript, it means a multi-index. In this spirit, for example, for $\mathbf{x} = (x_1, \dots, x_n)$, a monomial in variables x_1, \dots, x_n is denoted by $\mathbf{x}^{\mathbf{p}} = x_1^{p_1} \cdots x_n^{p_n}$. In addition, we also define $\mathbf{p}! = p_1! \cdots p_n!$ and $|\mathbf{p}| = p_1 + \cdots + p_n$; and if $\mathbf{p}, \mathbf{q} \in \mathbb{N}_0^n$, then we define $\delta_{\mathbf{p}, \mathbf{q}} = \delta_{p_1, q_1} \cdots \delta_{p_n, q_n}$.

Hypothesis 1. We will suppose in the remaining of the paper that the Lévy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$,

$$\int_{|\mathbf{x}| \geq \varepsilon} \exp(\lambda \|\mathbf{x}\|) \nu(d\mathbf{x}) < \infty.$$

This implies that

$$\int \mathbf{x}^{\mathbf{p}} \nu(d\mathbf{x}) < \infty, \quad |\mathbf{p}| \geq 2 \tag{1}$$

and that the characteristic function $E[\exp(i\theta \cdot X(t))]$ is analytic in a neighborhood of origin \mathbf{o} . As a consequence, $X(t)$ has moments of all orders and the polynomials are dense in $L^2(\mathbb{R}^n, \mathbb{P} \circ X(t)^{-1})$ for all $t > 0$.

Professor Nualart proposed author to use the following transformations of X which will play an important role in our analysis. We introduce power jump monomial processes of the form

$$X(t)^{(p_1, \dots, p_n)} \stackrel{\text{def}}{=} \sum_{0 < s \leq t} (\Delta X_1(s))^{p_1} \cdots (\Delta X_n(s))^{p_n},$$

The number $|p|$ is called the total degree of $X(t)^p$. Furthermore define

$$Y(t)^{(p_1, \dots, p_n)} \stackrel{\text{def}}{=} X(t)^{(p_1, \dots, p_n)} - \mathbb{E}[X(t)^{(p_1, \dots, p_n)}] = X(t)^{(p_1, \dots, p_n)} - m_p t,$$

the compensated power jump process of multi-index $p = (p_1, p_2, \dots, p_n)$. Under hypothesis 1, $Y(t)^{(p_1, \dots, p_n)}$ is a normal martingale, since for an integrable Lévy process Z , the process $\{Z_t - E[Z_t], t \geq 0\}$ is a martingale. We call $Y(t)^{(p_1, \dots, p_n)}$ the *Teugels martingale monomial* of multi-index (p_1, \dots, p_n) .

Remark 1. In the case of a Poisson process, all power jump processes will be the same, and equal to the original *Poisson process*. In the case of a *Brownian motion*, all power jump processes of order strictly greater than one will be equal to zero.

In the following we will introduce some concepts and basic properties of martingale polynomial. These new concepts and properties are totally similar to those of polynomial in n real variables (cf. Dunkl and Xu (2001)).

A *martingale polynomial* P in n Lévy variables $X = (X_1, X_2, \dots, X_n)$ is a linear combination of martingale monomials,

$$P(X) = \sum_{|p| \geq 1} c_p Y^p,$$

where the coefficients c_p are in the real numbers \mathbb{R} . The degree of a martingale polynomial is defined as the highest total degree of its martingale monomials. We shall use the abbreviation Π^n to denote the collection of all martingale polynomials in X . We also denote the space of martingale polynomials of degree at most d by Π_d^n . A martingale polynomial is called *homogeneous* if all the monomials appearing in it have the same total degree. Denote the space of homogeneous polynomials of degree $d \in \mathbb{N}$ in n variables by \mathcal{P}_d^n ; that is

$$\mathcal{P}_d^n = \left\{ P : P(X) = \sum_{|p|=d} c_p Y^p \right\}.$$

Every polynomial in Π^n can be written as a linear combination of homogeneous martingale polynomials; for $P \in \Pi_d^n$,

$$P(X) = \sum_{k=1}^d \sum_{|p|=k} c_p Y^p.$$

Denote by r_d^n the dimension of \mathcal{P}_d^n and it is well known that

$$r_d^n = \dim \mathcal{P}_d^n = \binom{d+n-1}{d} \quad \text{and} \quad \dim \Pi_d^n = \binom{d+n}{d} - 1.$$

We denote by \mathcal{N}^2 the space of one dimensional square integrable martingales M such that $\sup_t \mathbb{E}(M(t)^2) < \infty$, and $M(0) = 0$ a.s. Notice that if $M \in \mathcal{N}^2$, then $\lim_{t \rightarrow \infty} \mathbb{E}(M(t)^2) = \mathbb{E}(M(\infty)^2) < \infty$, and $M(t) = \mathbb{E}[M(\infty) | \mathcal{F}_t]$. Thus, each $M \in \mathcal{N}^2$ can be identified with its terminal value $M(\infty)$. As in Protter(2005, p.181), we say that two martingales $M, N \in \mathcal{N}^2$ are strongly orthogonal and we denote this by $M \times N$, if and only if the product MN is a uniformly integrable martingale. As noted in Protter (2005, p.181), one can prove that $M \times N$ if only if $[M, N]$ is a uniformly

integrable martingale. We say that two random vectors $X, Y \in L^2(\Omega, \mathcal{F})$ are weakly orthogonal, $X \perp Y$, if $E[XY] = 0$. Clearly, strong orthogonality implies weak orthogonality.

In the theory about orthogonal polynomials of several variables, we can apply the Gram-Schmidt process to the monomials with respect to the usual inner product to produce a sequence of orthogonal polynomials of several variables. Some details about the technique and theory of *orthogonal polynomials of several variables* refer to Dunkl and Xu (2001). In this paper, we shall apply the standard Gram-Schmidt process with the graded lexicographical order to generate a biorthogonal basis $\{H^p, p \in \mathbb{N}^n\}$, such that each $H^p(|p| = d)$ is a linear combination of the $Y^q \in \Pi_d^n$, with $|q| \leq |p|$ and the leading coefficient equal to 1. We set

$$H^p = Y^p + \sum_{q < p, |q|=|p|} c_q Y^q + \sum_{k=1}^{|p|-1} \sum_{|q|=k} c_q Y^q,$$

where $p = \{p_1, \dots, p_n\}$, $q = \{q_1, \dots, q_n\}$ and $<$ represent the relation of graded lexicographical order between two multi-indices.

We have that

$$\begin{aligned} [H^p, Y^q](t) &= \sum_{0 < s \leq t} (\Delta X_1(s))^{p_1+q_1} \dots (\Delta X_n(s))^{p_n+q_n} \\ &+ \sum_{0 < s \leq t} \sum_{1 \leq |\bar{p}| < |p|} c_{\bar{p}+q} (\Delta X_1(s))^{\bar{p}_1+q_1} \dots (\Delta X_n(s))^{\bar{p}_n+q_n} + \sum_{i=1}^n \sum_{j=1}^n c_{e_i+q} \sigma_{ij} t I_{\{q=e_j\}}. \end{aligned}$$

Let $m_{p+q} = \int \prod_{i=1}^n x_i^{p_i+q_i} \nu(dx)$, then we have that

$$\mathbb{E}[H^p, Y^q](t) = t \left(m_{p+q} + \sum_{1 \leq |\bar{p}| < |p|} c_{\bar{p}+q} m_{\bar{p}+q} + \sum_{i=1}^n \sum_{j=1}^n c_{e_i+q} \sigma_{ij} t I_{\{q=e_j\}} \right).$$

where e_i denotes the n -dimensional unit vector with i th component equal to one. In conclusion, we have that, $[H^p, Y^q]$ is a martingale if only if we have that $E[H^p, Y^q](1) = 0$.

Consider two spaces: The first space S_1 is defined as follows

$$\begin{aligned} S_1 &= \left\{ \sum_{k=1}^d \sum_{|p|=k} c_k(p_1, \dots, p_n) x_1^{p_1} \dots x_n^{p_n} + c_0 + \sum_{(i_1, \dots, i_n) \in \{0, -1\}^n, |i| \geq -(n-1)} c_{-1}(i_1, \dots, i_n) x_1^{i_1} \dots x_n^{i_n}; \right. \\ &\quad \left. d \in \{1, 2, \dots\}, c_j(p_1, \dots, p_n) \in \mathbb{R}, j = -1, 0, \dots, d; x_i \neq 0, i = 1, 2, \dots, n; i = (i_1, \dots, i_n) \right\} \end{aligned}$$

which is endowed with the scalar product $\langle \cdot, \cdot \rangle_1$ given by

$$\begin{aligned} \langle P(x), Q(x) \rangle_1 &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} P(x) Q(x) \prod_{i=1}^n x_i^2 \nu(dx) \\ &+ \sum_{i=1}^n \sum_{j=1}^n c_1 c_2 \sigma_{ij} I_{\{P(x)=c_1 x^{e_i-1}, Q(x)=c_2 x^{e_j-1}\}}, \quad \mathbf{1} = (1, 1, \dots, 1). \end{aligned}$$

Note that

$$\begin{aligned} &\langle x_1^{p_1-1} \dots x_n^{p_n-1}, x_1^{q_1-1} \dots x_n^{q_n-1} \rangle_1 \\ &= m_{p+q} + \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} I_{\{p=e_i, q=e_j\}}, \quad |p| \geq 1, \quad |q| \geq 1. \end{aligned}$$

Thus we can construct the other space S_2 which is the space of all linear transformations of the Teugels martingale monomials of the multivariate Lévy process, i.e.

$$\begin{aligned} S_2 &= \left\{ \sum_{p_1+\dots+p_n=d} c_d(p_1, \dots, p_n) Y(t)^{(p_1, \dots, p_n)} + \sum_{p_1+\dots+p_n=d-1} c_{d-1}(p_1, \dots, p_n) Y(t)^{(p_1, \dots, p_n)} \right. \\ &\quad \left. + \dots + \sum_{p_1+\dots+p_n=1} c_1(p_1, \dots, p_n) Y(t)^{(p_1, \dots, p_n)}, \quad d \geq 1, \quad i = 1, 2, \dots, n \right\}. \end{aligned}$$

We endow this space with the scalar product $\langle \cdot, \cdot \rangle_2$, given by

$$\begin{aligned} \langle Y^{(p_1, \dots, p_n)}, Y^{(q_1, \dots, q_n)} \rangle_2 &= E([Y^{(p_1, \dots, p_n)}, Y^{(q_1, \dots, q_n)}](1)) \\ &= m_{p+q} + \sum_{i=1}^n \sum_{j=1}^n c_{e_i+q} \sigma_{ij} I_{\{q=e_j\}}, \\ |p| &\geq 1, \quad |q| \geq 1. \end{aligned}$$

Because $\sum_{0 \leq s \leq t} (\Delta X_1(s))^{p_1} \cdots (\Delta X_n(s))^{p_n} \equiv \sum_{0 \leq s \leq t} (\Delta X_1(s))^{p_1} \cdots (\Delta X_n(s))^{p_n} I_{\cap_{i=1}^n (\Delta X_i(s) \neq 0)}$, one clearly sees that $x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} \leftrightarrow Y^{(p_1, \dots, p_n)}$ with $|p| \geq 1$ and is an isometry between S_1 and S_2 . An orthogonalization of $\{x_1^{-1} x_2^{-1} \cdots x_{n-1}^{-1}, x_1^{-1} x_3^{-1} \cdots x_n^{-1}, \dots, x_n^{-1}, 1, x_1, \dots, x_n, x_1^2, x_1 x_2, \dots, x_n^2, \dots\}$ in S_1 gives an orthogonalization of $\{Y^{(1,0,\dots,0)}, \dots, Y^{(0,\dots,0,1)}, Y^{(2,0,\dots,0)}, Y^{(1,1,0,\dots,0)}, \dots, Y^{(0,\dots,0,2)}, \dots\}$.

It is well known that *orthogonal polynomials of several variables are not unique* (cf. Dunkl and Xu (2001)). In the remaining of the paper, $\{H^p, p \in \mathbb{N}^n\}$ is a set of pairwise strongly orthogonal martingales given by the previous orthogonalization of $\{Y^p, p\}$. It is also worth to emphasis that all deduction procedures and results are the same once the orthogonal martingales are determinatively given.

3. Representation properties

3.1. Representation of a power of a Lévy process

For notation simplicity, here and hereafter we set

$$X_i^{(p_i)}(t) = \sum_{0 \leq s \leq t} (\Delta X_i(s))^{p_i}, \quad p_i \geq 2, \quad i = 1, 2, \dots, n.$$

and for convenience we put $X_i^{(1)}(t) = X_i(t)$. $p_i, q_i (i = 1, 2, \dots, n)$ are all some nonnegative integers. Note that not necessarily $X_i(t) = \sum_{0 \leq s \leq t} \Delta X_i(s)$ holds; it is only true in the bounded variation case with $\Sigma = O$. If $\Sigma = O$, clearly $[X_i, X_i](t) = X_i^{(2)}(t)$. The processes $X_i^{(p_i)} = \{X_i^{(p_i)}(t), t \geq 0\}$, $p_i = 1, 2, \dots$, are again Lévy processes. They jump at the same points as the original Lévy processes.

We have $E[X_i(t)] = E[X_i^{(1)}(t)] = tm_{i,1} < \infty$ and by Protter (1990,p29), that

$$E[X_i^{(p_i)}(t)] = E \left[\sum_{0 \leq s \leq t} (\Delta X_i(s))^{p_i} \right] = t \int x_i^{p_i} \nu(d\mathbf{x}) = m_{i,p_i} t < \infty, \quad p_i \geq 2.$$

Therefore, we can denote by

$$Y_i^{(p_i)}(t) \stackrel{\text{def}}{=} X_i^{(p_i)}(t) - E[X_i^{(p_i)}(t)] = X_i^{(p_i)}(t) - m_{i,p_i} t, \quad p_i = 1, 2, 3, \dots$$

the compensated power jump process of order p_i , and $Y_i^{(p_i)}$ is also a normal martingale.

We will express $(X_1(t+t_0) - X_1(t_0))^{k_1} (X_2(t+t_0) - X_2(t_0))^{k_2} \cdots (X_n(t+t_0) - X_n(t_0))^{k_n}$, $t_0, t \geq 0$, $k_i = 1, 2, \dots$, $i = 1, 2, \dots, n$, as a sum of stochastic integrals with respect to the special processes $Y_i^{(p_i)}(t)$, $i = 1, \dots, n$, $p_i = 1, \dots, k_i$.

Using Itô's formula(Protter, 1990, p.74, Theorem)we can write for $k_i \geq 2$, $i = 1, 2, \dots, n$,

$$\begin{aligned} & (X_1(t+t_0) - X_1(t_0))^{k_1} (X_2(t+t_0) - X_2(t_0))^{k_2} \cdots (X_n(t+t_0) - X_n(t_0))^{k_n} \\ &= \sum_{i=1}^n \int_0^t k_i (X_i(s+t_0) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0)) dX_i(s) \\ &+ \frac{1}{2} \sum_{i=1}^n \int_0^t \sigma_{ii}^2 k_i(k_i-1) (X_i(s+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} ds \\ &+ \sum_{1 \leq i < j \leq n} \int_0^t \sigma_{ij}^2 k_i k_j (X_i(s+t_0) - X_i(t_0))^{k_i-1} (X_j(s+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i,j} (X_\ell(s+t_0) - X_\ell(t_0))^{k_\ell} ds \\ &+ \sum_{0 \leq s \leq t} \left\{ \sum_{i=1}^n [(X_i(s+t_0) - X_i(t_0))^{k_i} - (X_i((s+t_0)-) - X_i(t_0))^{k_i}] \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \right. \\ &\quad \left. - \sum_{i=1}^n k_i (X_i((s+t_0)-) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \Delta X_i(s+t_0) \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \int_{t_0}^{t_0+t} k_i (X_i(u) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(u) - X_j(t_0)) dX_i^{(1)}(u) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sigma_{ii}^2 k_i (k_i - 1) \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(t+t_0) - X_j(t_0))^{k_j} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \right) \right] \\
&\quad + \sum_{1 \leq i < j \leq n} \sigma_{ij}^2 k_i k_j \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-1} (X_j(t+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(t+t_0) - X_\ell(t_0))^{k_\ell} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-1} (X_j(s+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(s+t_0) - X_\ell(t_0))^{k_\ell} ds \right) \right] \\
&\quad + \sum_{0 < s \leq t} \left\{ \sum_{i=1}^n [(X_i((s+t_0)-) + \Delta X_i(s+t_0) - X_i(t_0))^{k_i} - (X_i((s+t_0)-) - X_i(t_0))^{k_i}] \right. \\
&\quad \times \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \\
&\quad \left. - \sum_{i=1}^n k_i (X_i((s+t_0)-) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \Delta X_i(s+t_0) \right\} \\
&= \sum_{i=1}^n \int_{t_0}^{t_0+t} k_i (X_i(u) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(u) - X_j(t_0)) dX_i^{(1)}(u) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sigma_{ii}^2 k_i (k_i - 1) \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(t+t_0) - X_j(t_0))^{k_j} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \right) \right] \\
&\quad + \sum_{1 \leq i < j \leq n} \sigma_{ij}^2 k_i k_j \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-1} (X_j(t+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(t+t_0) - X_\ell(t_0))^{k_\ell} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-1} (X_j(s+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(s+t_0) - X_\ell(t_0))^{k_\ell} ds \right) \right] \\
&\quad + \sum_{0 < s \leq t} \sum_{i=1}^n \sum_{\ell=2}^{k_i} \binom{k_i}{\ell} (X_i((s+t_0)-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} (\Delta X_i(s+t_0))^\ell \\
&= \sum_{i=1}^n \int_{t_0}^{t_0+t} k_i (X_i(u) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(u) - X_j(t_0)) dX_i^{(1)}(u) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sigma_{ii}^2 k_i (k_i - 1) \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(t+t_0) - X_j(t_0))^{k_j} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \right) \right] \\
&\quad + \sum_{1 \leq i < j \leq n} \sigma_{ij}^2 k_i k_j \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-1} (X_j(t+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(t+t_0) - X_\ell(t_0))^{k_\ell} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-1} (X_j(s+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(s+t_0) - X_\ell(t_0))^{k_\ell} ds \right) \right] \\
&\quad + \sum_{t_0 < u \leq t+t_0} \sum_{i=1}^n \sum_{\ell=2}^{k_i} \binom{k_i}{\ell} (\Delta X_i(u))^\ell (X_i(u-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(u-) - X_j(t_0))^{k_j}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \sum_{\ell=2}^{k_i} \binom{k_i}{\ell} \int_{t_0}^{t+t_0} (X_i(u-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(u-) - X_j(t_0))^{k_j} dX_i^{(\ell)}(u) \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sigma_{ii}^2 k_i (k_i - 1) \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(t+t_0) - X_j(t_0))^{k_j} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \right) \right] \\
&\quad + \sum_{1 \leq i < j \leq n} \sigma_{ij}^2 k_i k_j \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-1} (X_j(t+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(t+t_0) - X_\ell(t_0))^{k_\ell} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-1} (X_j(s+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(s+t_0) - X_\ell(t_0))^{k_\ell} \right) ds \right]
\end{aligned} \tag{2}$$

Lemma 1. The power of an increment of a Lévy process, $(X_1(t+t_0) - X_1(t_0))^{k_1} (X_2(t+t_0) - X_2(t_0))^{k_2} \cdots (X_n(t+t_0) - X_n(t_0))^{k_n}$, has a representation of the form

$$\begin{aligned}
&(X_1(t+t_0) - X_1(t_0))^{k_1} (X_2(t+t_0) - X_2(t_0))^{k_2} \cdots (X_n(t+t_0) - X_n(t_0))^{k_n} \\
&= f^{(\mathbf{k})}(t, t_0) \\
&\quad + \sum_{m=1}^n \sum_{q_{i_1}=1}^{k_1} \cdots \sum_{q_{i_m}=1}^{k_m} \sum_{\substack{\times_{j=1}^m (p_{i_j,1}, \dots, p_{i_j, q_{i_j}}) \in \\ \times_{j=1}^m \{1, \dots, k_j\}^{q_{i_j}}}} \int_{t_0}^{t+t_0} \int_{t_0}^{t_{i_1,1}-} \cdots \int_{t_0}^{t_{i_1, q_{i_1}}-} \cdots \int_{t_0}^{t_{i_m,1}-} \cdots \\
&\quad \int_{t_0}^{t_{i_m, q_{i_m}}-} f_{(p_{i_1,1}, \dots, p_{i_1, q_{i_1}}; \dots; p_{i_m,1}, \dots, p_{i_m, q_{i_m}})}^{(\mathbf{k})}(t, t_0; t_{i_1,1}, \dots, t_{i_1, q_{i_1}}; \dots; t_{i_m,1}, \dots, t_{i_m, q_{i_m}}) \\
&\quad dY_{i_m}^{(p_{i_m, q_{i_m}})}(t_{i_m, q_{i_m}}) \cdots dY_{i_m}^{(p_{i_m, 2})}(t_{i_m, 2}) dY_{i_m}^{(p_{i_m, 1})}(t_{i_m, 1}) \\
&\quad \cdots dY_{i_1}^{(p_{i_1, q_{i_1}})}(t_{i_1, q_{i_1}}) \cdots dY_{i_1}^{(p_{i_1, 2})}(t_{i_1, 2}) dY_{i_1}^{(p_{i_1, 1})}(t_{i_1, 1})
\end{aligned} \tag{3}$$

where the $f_{(p_{i_1,1}, \dots, p_{i_1, q_{i_1}}; \dots; p_{i_m,1}, \dots, p_{i_m, q_{i_m}})}^{(\mathbf{k})}(t, t_0; t_{i_1,1}, \dots, t_{i_1, q_{i_1}}; \dots; t_{i_m,1}, \dots, t_{i_m, q_{i_m}})$ are deterministic functions in $L^2(\mathbb{R}_+^{q_{i_1} + \dots + q_{i_m}})$, and $\mathbf{k} = (k_1, \dots, k_n)$. In addition, the index m controls the number of $Y_i(t)$, $i = 1, \dots, n$, chosen from $Y_1(t), \dots, Y_n(t)$. After the m is fixed, (i_1, \dots, i_m) indicates an arbitrary subset of the integer set $(1, 2, \dots, n)$. After the (i_1, \dots, i_m) is fixed, for $i_j \in (i_1, \dots, i_m)$, the $(t_{i_j,1}, \dots, t_{i_j, q_{i_j}})$ with double index (i_j, \cdot) indicates the time-points chosen from the time-points (t_1, \dots, t_{k_j}) corresponding to $(Y_{i_j}(t_1), \dots, Y_{i_j}(t_{k_j}))$. The meaning of power index $p_{i_j, \cdot}$ is similar.

Proof Representation (3) follows from (2), where we bring in the right compensations, i.e. we can write

$$\begin{aligned}
&\sum_{i=1}^n \sum_{\ell=2}^{k_i} \binom{k_i}{\ell} \int_{t_0}^{t+t_0} (X_i(s-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s-) - X_j(t_0))^{k_j} dX_i^{(\ell)}(s) \\
&= \sum_{i=1}^n \sum_{\ell=2}^{k_i} \binom{k_i}{\ell} \int_{t_0}^{t+t_0} (X_i(s-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s-) - X_j(t_0))^{k_j} dY_i^{(\ell)}(s) \\
&\quad + \sum_{i=1}^n \sum_{\ell=1}^{k_i} \binom{k_i}{\ell} m_{i\ell} \int_{t_0}^{t+t_0} (X_i(s-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s-) - X_j(t_0))^{k_j} ds \\
&= \sum_{i=1}^n \sum_{\ell=1}^{k_i} \binom{k_i}{\ell} \int_{t_0}^{t+t_0} (X_i(s-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s-) - X_j(t_0))^{k_j} dY_i^{(\ell)}(s) \\
&\quad + \sum_{i=1}^n \sum_{\ell=1}^{k_i-1} \binom{k_i}{\ell} m_{i\ell} t (X_i(t+t_0) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(t+t_0) - X_j(t_0))^{k_j} ds \\
&\quad - \sum_{i=1}^n \sum_{\ell=1}^{k_i-1} \binom{k_i}{\ell} m_{i\ell} \int_{t_0}^{t+t_0} sd \left((X_i(s-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s-) - X_j(t_0))^{k_j} \right) \\
&\quad + \sum_{i=1}^n m_{ik_i} t.
\end{aligned} \tag{4}$$

Combining (2) and (4) gives

$$\begin{aligned}
& \prod_{i=1}^n (X_i(t+t_0) - X_i(t_0))^{k_i} \\
&= \frac{1}{2} \sum_{i=1}^n \sigma_{ii}^2 k_i (k_i - 1) \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(t+t_0) - X_j(t_0))^{k_j} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-2} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \right) \right] \\
&\quad + \sum_{1 \leq i < j \leq n} \sigma_{ij}^2 k_i k_j \left[t (X_i(t+t_0) - X_i(t_0))^{k_i-1} (X_j(t+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(t+t_0) - X_\ell(t_0))^{k_\ell} \right. \\
&\quad \left. - \int_0^t sd \left((X_i(s+t_0) - X_i(t_0))^{k_i-1} (X_j(s+t_0) - X_j(t_0))^{k_j-1} \prod_{\ell \neq i, j} (X_\ell(s+t_0) - X_\ell(t_0))^{k_\ell} \right) ds \right] \\
&\quad + \sum_{i=1}^n \sum_{\ell=1}^{k_i} \binom{k_i}{\ell} \int_{t_0}^{t+t_0} (X_i(s-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s-) - X_j(t_0))^{k_j} dY_i^{(\ell)}(s) \\
&\quad + \sum_{i=1}^n \sum_{\ell=1}^{k_i-1} \binom{k_i}{\ell} m_{i\ell} t (X_i(t+t_0) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(t+t_0) - X_j(t_0))^{k_j} ds \\
&\quad - \sum_{i=1}^n \sum_{\ell=1}^{k_i-1} \binom{k_i}{\ell} m_{i\ell} \int_{t_0}^{t+t_0} sd \left((X_i(s-) - X_i(t_0))^{k_i-\ell} \prod_{j \neq i} (X_j(s-) - X_j(t_0))^{k_j} \right) \\
&\quad + \sum_{i=1}^n m_{ik_i} t.
\end{aligned} \tag{5}$$

The last equation is in terms of powers of increments of X_i which are strictly lower than k_i . So by induction representation (3) can be proved. \square

Notice that taking the expectation in (3) yields

$$E \left[\prod_{i=1}^n (X_i(t+t_0) - X_i(t_0))^{k_i} \right] = f^{(k)}(t, t_0) = f^{(k)}(t), \quad t, t_0 \geq 0,$$

which is independent of t_0 .

Moreover, it can easily be seen that

$$f_{(p_{i_1,1}, \dots, p_{i_1, q_{i_1}}; \dots; p_{i_m,1}, \dots, p_{i_m, q_{i_m}})}^{(k)}(t, t_0; t_{i_1,1}, \dots, t_{i_1, q_{i_1}}; \dots; t_{i_m,1}, \dots, t_{i_m, q_{i_m}})$$

are just real multivariate polynomials of degree less than k_i and that we have

$$f_{(p_{i_1,1}, \dots, p_{i_1, q_{i_1}}; \dots; p_{i_m,1}, \dots, p_{i_m, q_{i_m}})}^{(k)}(t, t_0; t_{i_1,1}, \dots, t_{i_1, q_{i_1}}; \dots; t_{i_m,1}, \dots, t_{i_m, q_{i_m}}) = 0,$$

whenever $p_{i_1, q_{i_1}} + \dots + p_{i_j, q_{i_j}} > k_j$.

Because we can switch by a linear transformation from the $Y_{i_m}^{(p_{i_m,2})}(t_{i_m,2})$ to $H_{i_m}^{(p_{i_m,2})}(t_{i_m,2})$, it is clear that we also proved the next representation.

Lemma 2. *The power of an increment of a Lévy process, $(X_1(t+t_0) - X_1(t_0))^{k_1} (X_2(t+t_0) - X_2(t_0))^{k_2} \dots (X_n(t+t_0) - X_n(t_0))^{k_n}$, has a representation of the form*

$$\begin{aligned}
& (X_1(t+t_0) - X_1(t_0))^{k_1} (X_2(t+t_0) - X_2(t_0))^{k_2} \dots (X_n(t+t_0) - X_n(t_0))^{k_n} \\
&= f^{(k)}(t, t_0) + \sum_{d=1}^{|k|} \sum_{\mathbf{p}_1 \in \mathbb{N}_d^n} \int_{t_0}^{t+t_0} h_{(\mathbf{p}_1)}^{(k)}(t, t_0; t_1) dH^{\mathbf{p}_1}(t_1) \\
&\quad + \sum_{d=1}^{|k|} \sum_{\mathbf{p}_1 + \mathbf{p}_2 \in \mathbb{N}_d^n} \int_{t_0}^{t+t_0} \int_{t_0}^{t_1} h_{(\mathbf{p}_1, \mathbf{p}_2)}^{(k)}(t, t_0; t_1, t_2) dH^{\mathbf{p}_2}(t_2) dH^{\mathbf{p}_1}(t_1) \\
&\quad + \dots \\
&\quad + \sum_{d=1}^{|k|} \sum_{\mathbf{p}_1 + \dots + \mathbf{p}_{|k|} \in \mathbb{N}_d^n} \int_{t_0}^{t+t_0} \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{|k|-1}} h_{(\mathbf{p}_1, \dots, \mathbf{p}_{|k|})}^{(k)}(t, t_0; t_1, t_2, \dots, t_{|k|}) dH^{\mathbf{p}_{|k|}}(t_{|k|}) \dots dH^{\mathbf{p}_2}(t_2) dH^{\mathbf{p}_1}(t_1) \\
&= \sum_{m=1}^{|k|} \sum_{\mathbf{p}_1 + \dots + \mathbf{p}_m \in \mathbb{N}_d^n} \int_{t_0}^{t+t_0} \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{m-1}} h_{(\mathbf{p}_1, \dots, \mathbf{p}_m)}^{(k)}(t, t_0; t_1, t_2, \dots, t_m) dH^{\mathbf{p}_m}(t_m) \dots dH^{\mathbf{p}_2}(t_2) dH^{\mathbf{p}_1}(t_1)
\end{aligned}$$

where the index m controls the number of integral variables in each multiple integral, $\mathbf{p}_i = (p_{i_1}, \dots, p_{i_n}) \in \mathbb{N}_0^n$, $\mathbb{N}_d^n = \{\mathbf{p} \in \mathbb{N}_0^n : |\mathbf{p}| = d\}$ and $h_{(\mathbf{p}_1, \dots, \mathbf{p}_m)}^{(k)}(t, t_0; t_1, t_2, \dots, t_m)$ are deterministic functions in $L^2(\mathbb{R}_+^m)$. Other notations are the same as Lemma 1.

3.2. Representation of a square integrable random variable

We first recall that $\{H_i^{(p_i)}, i = 1, 2, \dots, n; p_i = 1, 2, \dots\}$ is a set of pairwise strongly orthogonal martingales, obtained by the orthogonalization procedure described at the end of Section 2.

We denote by

$$\mathcal{H}^{(\mathbf{p}_1, \dots, \mathbf{p}_m)} = \left\{ F \in L^2(\Omega) : \int_{t_0}^{t+t_0} \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{m-1}} h_{(\mathbf{p}_1, \dots, \mathbf{p}_m)}^{(k)}(t, t_0; t_1, t_2, \dots, t_m) dH^{\mathbf{p}_m}(t_m) \dots dH^{\mathbf{p}_2}(t_2) dH^{\mathbf{p}_1}(t_1) \right. \\ \left. \mathbf{p}_j \in \mathbb{N}^n, \quad j = 1, 2, \dots, m \right\} \quad (6)$$

We say that two multi-indexes

$$(\mathbf{p}_1, \dots, \mathbf{p}_m) \quad \text{and} \quad (\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{\tilde{m}})$$

are different if $m \neq \tilde{m}$ or when $m \equiv \tilde{m}$, if there exists a subindex $1 \leq l \leq m = \tilde{m}$, such that $\mathbf{p}_l \neq \tilde{\mathbf{p}}_l$, and denote this by

$$(\mathbf{p}_1, \dots, \mathbf{p}_m) \neq (\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{\tilde{m}})$$

Proposition 1. *If*

$$(\mathbf{p}_1, \dots, \mathbf{p}_m) \neq (\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{\tilde{m}})$$

then

$$\mathcal{H}^{(\mathbf{p}_1, \dots, \mathbf{p}_m)} \perp \mathcal{H}^{(\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{\tilde{m}})}.$$

Proof Suppose we have two random variables $K \in \mathcal{H}^{(\mathbf{p}_1, \dots, \mathbf{p}_m)}$ and $L \in \mathcal{H}^{(\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{\tilde{m}})}$. We need to prove that if

$$(\mathbf{p}_1, \dots, \mathbf{p}_m) \neq (\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_{\tilde{m}})$$

then $K \perp L$.

For the case $m = \tilde{m}$, we use induction on m . Take first $m = \tilde{m} = 1$ and assume the following representations for K and L :

$$K = \int_0^\infty f(t_1) dH^{\mathbf{p}_1}(t_1), \quad L = \int_0^\infty g(t_1) dH^{\tilde{\mathbf{p}}_1}(t_1)$$

where we must have $\mathbf{p}_1 \neq \tilde{\mathbf{p}}_1$. By construction $H^{\mathbf{p}_1}$ and $H^{\tilde{\mathbf{p}}_1}$ are strongly orthogonal martingales. Using the fact that stochastic integrals with respect to strongly orthogonal martingales are again strongly orthogonal (Protter, 1990, Lemma 2 and Theorem 35, p.149) and thus also weakly orthogonal, it immediately follows that $K \perp L$.

Suppose the theorem holds for all $1 \leq m = \tilde{m} \leq n-1$. We are going to prove the theorem for $m = \tilde{m} = n$. Assume the following representations:

$$K = \int_{t_0}^{t+t_0} \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{m-1}} h_{(\mathbf{p}_1, \dots, \mathbf{p}_m)}^{(k)}(t, t_0; t_1, t_2, \dots, t_m) dH^{\mathbf{p}_m}(t_m) \dots dH^{\mathbf{p}_2}(t_2) dH^{\mathbf{p}_1}(t_1) \\ = \int_0^\infty \alpha(t_1) dH^{\mathbf{p}_1}(t_1)$$

$$L = \int_{t_0}^{t+t_0} \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{m-1}} g_{(\tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_m)}^{(k)}(t, t_0; t_1, t_2, \dots, t_m) dH^{\tilde{\mathbf{p}}_m}(t_m) \dots dH^{\tilde{\mathbf{p}}_2}(t_2) dH^{\tilde{\mathbf{p}}_1}(t_1) \\ = \int_0^\infty \beta(t_1) dH^{\tilde{\mathbf{p}}_1}(t_1)$$

There are two possibilities: (1) $\mathbf{p}_1 = \tilde{\mathbf{p}}_1$ and (2) $\mathbf{p}_1 \neq \tilde{\mathbf{p}}_1$. In the former case we must have that

$$(\mathbf{p}_2, \dots, \mathbf{p}_m) \neq (\tilde{\mathbf{p}}_2, \dots, \tilde{\mathbf{p}}_{\tilde{m}})$$

and thus by induction $\alpha(t_1) \perp \beta(t_1)$, so that

$$\begin{aligned} E[KL] &= E \left[\int_0^\infty \alpha_s \beta_s ds < H^{p_1}, H^{p_1} >_s \right] \\ &= \int_0^\infty E(\alpha_s \beta_s) ds < H^{p_1}, H^{p_1} >_s = 0. \end{aligned}$$

In the latter case we use again the fact that stochastic integrals with respect to strongly orthogonal martingales are again strongly orthogonal (Protter, 1990, Lemma 2 and Theorem 35, p.149) and thus also weakly orthogonal. So it immediately follows that $K \perp L$.

For the case $m \neq \tilde{m}$, a similar argument can be used together with the fact that all elements of every $\mathcal{H}^{(p_1, \dots, p_\ell)}$, $\ell \geq 1$, have mean zero and thus are orthogonal w.r.t. the constants. \square

Proposition 2. *Let*

$$\mathcal{P} = \left\{ \prod_{i=1}^n \prod_{j=1}^m (X_i(t_j) - X_i(t_{j-1}))^{k_{i,j}} : m \geq 0, 0 \leq t_0 \leq t_1 < t_2 < \dots < t_m, k_{1,1}, \dots, k_{n,m} \geq 1 \right\}$$

then we have that \mathcal{P} is a total family in $L^2(\Omega, \mathcal{F})$, i.e. the linear subspace spanned by \mathcal{P} is dense in $L^2(\Omega, \mathcal{F})$.

Proof Let $Z \in L^2(\Omega, \mathcal{F})$ and $Z \perp \mathcal{P}$. For any given $\varepsilon > 0$, there exists a finite set $\{0 < s_1 < \dots < s_m\}$ and a square integrable random variable $Z_\varepsilon \in L^2(\Omega, \sigma(X(s_1), X(s_2), \dots, X(s_m)))$ such that

$$E[\|Z - Z_\varepsilon\|^2] < \varepsilon.$$

So there exists a Borel function f such that

$$Z_\varepsilon = f_\varepsilon(X(s_1), X(s_2) - X(s_1), \dots, X(s_m) - X(s_{m-1})).$$

Because the polynomials are dense in $L^2(\mathbb{R}^n, \mathbb{P} \circ X(t)^{-1})$ for each $t > 0$, we can approximate Z_ε by polynomials. Furthermore because $Z \perp \mathcal{P}$, we have $E[ZZ_\varepsilon] = 0$. Then

$$E[\|Z\|^2] = E[Z \cdot (Z - Z_\varepsilon)] \leq \sqrt{E[\|Z\|^2]} E[\|Z - Z_\varepsilon\|^2] \leq \sqrt{\varepsilon E[\|Z\|^2]},$$

and Letting $\varepsilon \rightarrow 0$ yields $Z = \mathbf{0}$ a.s. Thus \mathcal{P} is a total family in $L^2(\Omega, \mathcal{F})$. \square

We are now in a position to prove our main theorem.

Theorem 1. *(Chaotic representation property (CRP)). Every random variable F in $L^2(\Omega, \mathcal{F})$ has a representation of the form*

$$\begin{aligned} F &= \mathbb{E}(F) + \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \sum_{p_1 + \dots + p_m \in \mathbb{N}_d^n} \int_0^\infty \int_0^{t_1} \dots \\ &\quad \int_0^{t_{m-1}} f_{(p_1, \dots, p_m)}^{(k)}(t, t_0; t_1, t_2, \dots, t_m) dH^{p_m}(t_m) \dots dH^{p_2}(t_2) dH^{p_1}(t_1) \end{aligned}$$

where the $f_{(p_1, \dots, p_m)}^{(k)}(t, t_0; t_1, t_2, \dots, t_m)$'s are functions in $L^2(\mathbb{R}_+^m)$.

Proof Because \mathcal{P} is a total family in $L^2(\Omega, \mathcal{F})$, it is sufficient to prove that every element of \mathcal{P} has a representation of the desired form. This follows from the fact that \mathcal{P} is build up from terms of the form $\prod_{i=1}^n \prod_{j=1}^m (X_i(t_j) - X_i(t_{j-1}))^{k_{i,j}}$, wherein every term has on its turn a representation of the form (6), and we can nicely combine two terms in the desired representation. Indeed, we have for all $k_i, l_i \geq 1$, $i = 1, \dots, n$, and $0 \leq t < s \leq u < v$, that the product of $\prod_{i=1}^n (X_i(s) - X_i(t))^{k_i} (X_i(v) - X_i(u))^{l_i}$ is a sum of products of the form AB where

$$A = \int_t^s \int_t^{t_1} \dots \int_t^{t_{m-1}} h_{(p_1, \dots, p_m)}^{(k)}(s, t; t_1, t_2, \dots, t_m) dH^{p_m}(t_m) \dots dH^{p_2}(t_2) dH^{p_1}(t_1)$$

and

$$B = \int_u^v \int_u^{u_1^-} \cdots \int_u^{u_{\bar{m}-1}^-} h_{(\bar{p}_1, \dots, \bar{p}_{\bar{m}})}^{(L)}(v, u; u_1, u_2, \dots, u_{\bar{m}}) dH^{\bar{p}_{\bar{m}}}(t_{\bar{m}}) \cdots dH^{\bar{p}_2}(t_2) dH^{\bar{p}_1}(t_1)$$

where m and \bar{m} are two integers.

We can write

$$\begin{aligned} AB &= \int_u^v \int_u^{u_1^-} \cdots \int_u^{u_{\bar{m}-1}^-} \int_t^s \int_t^{t_1^-} \cdots \int_t^{t_{m-1}^-} h_{(\bar{p}_1, \dots, \bar{p}_{\bar{m}})}^{(L)}(v, u; u_1, u_2, \dots, u_{\bar{m}}) \\ &\quad h_{(p_1, \dots, p_m)}^{(k)}(s, t; t_1, t_2, \dots, t_m) dH^{p_m}(t_m) \cdots dH^{p_2}(t_2) dH^{p_1}(t_1) dH^{\bar{p}_{\bar{m}}}(t_{\bar{m}}) \cdots dH^{\bar{p}_2}(t_2) dH^{\bar{p}_1}(t_1) \\ &= \int_0^\infty \int_0^{t_1^-} \cdots \int_0^{u_{\bar{m}-1}^-} \int_0^{u_{\bar{m}}^-} \int_0^{t_1^-} \cdots \int_0^{t_{m-1}^-} 1_{(u, v]}(u_1) 1_{(u, u_1]}(u_2) \cdots 1_{(u, u_{\bar{m}-1}]}(u_{\bar{m}}) \\ &\quad 1_{(t, s]}(t_1) 1_{(t, t_1]}(t_2) \cdots 1_{(t, t_{m-1}]}(t_m) h_{(\bar{p}_1, \dots, \bar{p}_{\bar{m}})}^{(L)}(v, u; u_1, u_2, \dots, u_{\bar{m}}) h_{(p_1, \dots, p_m)}^{(k)}(s, t; t_1, t_2, \dots, t_m) \\ &\quad dH^{p_m}(t_m) \cdots dH^{p_2}(t_2) dH^{p_1}(t_1) dH^{\bar{p}_{\bar{m}}}(t_{\bar{m}}) \cdots dH^{\bar{p}_2}(t_2) dH^{\bar{p}_1}(t_1) \end{aligned}$$

and the desired representation follows. \square

Theorem 2. (Predictable representation property (PRP)). Every random variable F in $L^2(\Omega, \mathcal{F})$ has a representation of the form

$$F = \mathbb{E}(F) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^\infty \Phi^p(s) dH^p(t_m)(s)$$

where $\Phi^p(s)$ is predictable.

Proof From the above theorem, we know that F has a representation of the form

$$\begin{aligned} &F - \mathbb{E}(F) \\ &= \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \sum_{p_1 + \dots + p_m \in \mathbb{N}_d^n} \int_0^\infty \int_0^{t_1^-} \cdots \int_0^{t_{m-1}^-} f_{(p_1, \dots, p_m)}^{(k)}(t_1, t_2, \dots, t_m) dH^{p_m}(t_m) \cdots dH^{p_2}(t_2) dH^{p_1}(t_1) \\ &= \sum_{d=1}^{\infty} \sum_{p_1 \in \mathbb{N}_d^n} \int_0^\infty f_{(p_1)}^{(k)}(t_1) dH^{p_1}(t_1) + \sum_{d=1}^{\infty} \sum_{p_1 \in \mathbb{N}_d^n} \int_0^\infty \left[\sum_{k=0}^\infty \sum_{p_2 + \dots + p_m \in \mathbb{N}_k^n} \int_0^{t_1^-} \cdots \right. \\ &\quad \left. \int_0^{t_{m-1}^-} f_{(p_1, \dots, p_m)}^{(k)}(t_1, t_2, \dots, t_m) dH^{p_m}(t_m) \cdots dH^{p_2}(t_2) \right] dH^{p_1}(t_1) \\ &= \sum_{d=1}^{\infty} \sum_{p_1 \in \mathbb{N}_d^n} \int_0^\infty \left[f_{(p_1)}^{(k)}(t_1) \right. \\ &\quad \left. + \sum_{k=0}^\infty \sum_{p_2 + \dots + p_m \in \mathbb{N}_k^n} \int_0^{t_1^-} \int_0^{t_{m-1}^-} f_{(p_1, \dots, p_m)}^{(k)}(t_1, t_2, \dots, t_m) dH^{p_m}(t_m) \cdots dH^{p_2}(t_2) \right] dH^{p_1}(t_1) \\ &= \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^\infty \Phi^p(s) dH^p(t_m)(s) \end{aligned}$$

which is exactly of the form we want. \square

Remark 2. Because we can identify every martingale $M \in \mathcal{U}^2$ with its terminal value $M_\infty \in L^2(\Omega, \mathcal{F})$ and because $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$, we have the predictable representation

$$M_t = \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_d^n} \int_0^t \Phi^p(s) dH^p(t_m)(s)$$

which is a sum of strongly orthogonal martingales.

Another consequence of the chaotic representation property, is the following theorem:

Theorem 3. We have the following space decomposition:

$$L^2(\Omega, \mathcal{F}) = \mathbb{R} \oplus \left(\bigoplus_{d=1}^{\infty} \bigoplus_{p \in \mathbb{N}_d^n} \mathcal{H}^p \right).$$

Remark 3. The Lévy-Khintchine formula has a simpler expression when the sample paths of the related Lévy process have bounded variation on every compact time interval a.s. It is well known (Bertoin, 1996, p.15), that a Lévy process has bounded variation if and only if $\Sigma = \mathbf{0}$, and $\int (1 \wedge \|\mathbf{x}\|) \nu(d\mathbf{x}) < \infty$. In that case the characteristic exponent can be re-expressed as

$$\psi(\boldsymbol{\theta}) = i\mathbf{d} \cdot \boldsymbol{\theta} + \int_{\mathbb{R}^n} (\exp(i\boldsymbol{\theta} \cdot \mathbf{x}) - 1) \nu(d\mathbf{x}).$$

Furthermore, we can write

$$X_i(t) = dt + \sum_{0 < s \leq t} \Delta X_i(s), \quad t \geq 0, \quad i = 1, 2, \dots, n. \quad (7)$$

and the calculations simplify somewhat because $\Sigma = \mathbf{0}$ and for $k \geq 1$,

$$\begin{aligned} & \sum_{i=1}^n \int_0^t k_i(X_i(s+t_0) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0)) dX_i(s) \\ = & \sum_{0 < s \leq t} \left\{ \sum_{i=1}^n k_i(X_i((s+t_0)-) - X_i(t_0))^{k_i-1} \prod_{j \neq i} (X_j(s+t_0) - X_j(t_0))^{k_j} \Delta X_i(s+t_0) \right\} \end{aligned}$$

4. Examples

Multidimensional models with jumps are more difficult to construct than one-dimensional ones. A simple method to introduce jumps into a multidimensional model is to take a multivariate Brownian motion and time change it with a univariate subordinator (refer to Cont and Tankov (2004)). The multidimensional versions of the models include variance gamma, normal inverse Gaussian and generalized hyperbolic processes. The principal advantage of this method is its simplicity and analytic tractability; in particular, processes of this type are easy to simulate. Another method to introduce jumps into a multidimensional model is so-called method of Lévy copulas proposed by Kallsen and Tankov (2006). The principle advantage in this way lies in that the dependence among components of the multidimensional Lévy processes can be completely characterized with a Lévy copula. This allows us to give a systematic method to construct multidimensional Lévy processes with specified dependence.

In the following first and third examples, we define a multivariate gamma process and a multivariate Meixner process by using Lévy copulas, and furthermore discuss their orthogonalization procedures. All the concepts and notations are adopted from the Kallsen and Tankov (2006). In particular, for $n \geq 2$, the Lévy copula $F(u_1, \dots, u_n) : \mathbb{R}^n \rightarrow \mathbb{R}$ is taken as

$$F(u_1, \dots, u_n) = 2^{2-n} \left(\sum_{j=1}^n |u_j|^{-\theta} \right)^{-1/\theta} (\eta I_{\{u_1 \dots u_n \geq 0\}} - (1-\eta) I_{\{u_1 \dots u_n < 0\}}). \quad (8)$$

It defines a two parameter family of Lévy copulas which resembles the Clayton family of ordinary copulas. It is in fact a Lévy copula homogeneous of order 1, for any $\theta > 0$ and any $\eta \in [0, 1]$.

In addition, we know that if the tail integrals $U_i(x_i)$, $i = 1, \dots, n$, are absolutely continuous, we can compute the Lévy density of the Lévy copula process by differentiation as follows:

$$\nu(dx_1, \dots, dx_n) = \partial_1 \dots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \nu_1(dx_1) \dots \nu_n(dx_n) \quad (9)$$

where $\nu_1(dx_1), \dots, \nu_n(dx_n)$ are marginal Lévy densities.

4.1. The multivariate gamma process

In the literature, the multivariate gamma distributions on \mathbb{R}^n have several non-equivalent definitions (refer to Johnson and Balakrishnan (1997)). Here we consider only a multivariate gamma process by using copula. The multivariate

Gamma process $\mathbf{G}(t) = (G_1(t), G_2(t), \dots, G_n(t))^T$ is a multivariate Lévy process with the marginal distribution density functions of $G_i(t)$, $i = 1, 2, \dots, n$ given by

$$f_{G_i(t)}(x_i) = \frac{1}{\Gamma(\gamma_i t)} \lambda_i^{\gamma_i t} x_i^{\gamma_i t - 1} \exp\{-\lambda_i x_i\},$$

$$x_i > 0, \quad \lambda_i, \gamma_i > 0 \quad i = 1, 2, \dots, n.$$

The corresponding marginal characteristic functions are given by

$$\mathbb{E}(e^{i\theta_i G_i(t)}) = \left(1 - \frac{i\theta_i}{\lambda_i}\right)^{-\gamma_i t}, \quad i = 1, 2, \dots, n.$$

The corresponding marginal Lévy measures are given by

$$\nu_i(dx_i) = \frac{\gamma_i}{x_i} \exp\{-\lambda_i x_i\} I_{(0, \infty)}(x_i), \quad i = 1, 2, \dots, n$$

The corresponding Lévy measure is given by

$$\nu_P(d\mathbf{x}) = \partial_1 \cdots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n x_i} \exp\{-\sum_{i=1}^n \lambda_i x_i\} I_{(0, \infty)^n}(x_1, \dots, x_n),$$

where $d\mathbf{x} = dx_1 \cdots dx_n$. The n-dimensional Gamma processes are used i.a. in insurance mathematics (Dickson and Waters, 1993, 1996; Dufresne and Gerber, 1993; Dufresne et al., 1991).

We denote by

$$G_i^{(p_i)}(t) = \sum_{0 < s \leq t} (\Delta G_i(s))^{p_i}, \quad p_i \geq 1, \quad i = 1, 2, \dots, n$$

the power jump processes of $G_i(t)$. In addition, set $\mathbf{G}^{(p_1, \dots, p_n)}(t) = (G_1^{(p_1)}, \dots, G_n^{(p_n)})$, where $(p_1, \dots, p_n) \in \mathbb{N}^n$. Using the exponential formula (Bertoin, 1996), and the change of the variable $\mathbf{z} = (x_1^{p_1}, \dots, x_n^{p_n})$, we obtain for $p_1 + \dots + p_n \geq 1$

$$\begin{aligned} & E \left[\exp \left(i\theta^T \mathbf{G}^{(p_1, \dots, p_n)}(t) \right) \right] \\ &= \exp \left(t \int_{\mathbb{R}_+^n} \left(\exp \left(i \sum_{i=1}^n \theta_i x_i^{p_i} \right) - 1 \right) \partial_1 \cdots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n x_i} \exp\{-\sum_{i=1}^n \lambda_i x_i\} d\mathbf{x} \right) \\ &= \exp \left(t \int_{\mathbb{R}_+^n} \left(\exp(i\theta^T \mathbf{z}) - 1 \right) \partial_1 \cdots \partial_n F|_{\xi_1=U_1(z_1^{1/p_1}), \dots, \xi_n=U_n(z_n^{1/p_n})} \frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n p_i z_i} \exp\{-\sum_{i=1}^n \lambda_i z_i^{1/p_i}\} d\mathbf{z} \right) \end{aligned}$$

which means that the Lévy measure of $\mathbf{G}^{(p_1, \dots, p_n)}$ is

$$\partial_1 \cdots \partial_n F|_{\xi_1=U_1(z_1^{1/p_1}), \dots, \xi_n=U_n(z_n^{1/p_n})} \frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n p_i z_i} \exp\{-\sum_{i=1}^n \lambda_i z_i^{1/p_i}\} d\mathbf{z}$$

Introduce power jump processes of the form

$$G(t)^{(p_1, \dots, p_n)} \stackrel{\text{def}}{=} \sum_{0 < s \leq t} (\Delta G_1(s))^{p_1} \cdots (\Delta G_n(s))^{p_n}$$

and then define the Teugels martingale monomial

$$\hat{G}(t)^{(p_1, \dots, p_n)} \stackrel{\text{def}}{=} G(t)^{(p_1, \dots, p_n)} - \mathbb{E}[G(t)^{(p_1, \dots, p_n)}] = G(t)^{(p_1, \dots, p_n)} - m_P t.$$

Because

$$\begin{aligned} & E \left[\sum_{0 < s \leq t} (\Delta G_1(s))^{p_1} \cdots (\Delta G_n(s))^{p_n} \right] \\ &= t \int_{\mathbb{R}_+^n} x_1^{p_1} \cdots x_n^{p_n} \partial_1 \cdots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \prod_{i=1}^n \frac{\gamma_i}{x_i} \exp\left\{-\sum_{i=1}^n \lambda_i x_i\right\} dx, \quad |p| \geq 1, \end{aligned}$$

Next, we orthogonalize the set \hat{G}^p of martingales. So we are looking for a set of martingales

$$H^p = \hat{G}^p + \sum_{q < p, |q|=|p|} c_q \hat{G}^q + \sum_{k=1}^{|p|-1} \sum_{|q|=k} c_q \hat{G}^q, \quad (10)$$

such that H^p is strongly orthogonal to $H^{\tilde{p}}$, for $p \neq \tilde{p}$.

The first space S_1 in the gamma case is defined as follows

$$\begin{aligned} S_1 = & \left\{ \sum_{k=1}^d \sum_{|p|=k} c_k(p_1, \dots, p_n) x_1^{p_1} \cdots x_n^{p_n} + c_0 + \sum_{(i_1, \dots, i_n) \in \{0, -1\}^n, |i| \geq -(n-1)} c_{-1}(i_1, \dots, i_n) x_1^{i_1} \cdots x_n^{i_n}; \right. \\ & \left. d \in \{1, 2, \dots\}, c_j(p_1, \dots, p_n) \in \mathbb{R}, j = -1, 0, \dots, d; x_i > 0, i = 1, \dots, n; i = (i_1, \dots, i_n) \right\} \end{aligned}$$

which is endowed with a scalar product $\langle \cdot, \cdot \rangle_1$, given by

$$\begin{aligned} & \langle P(x), Q(x) \rangle_1 \\ &= \int_0^{+\infty} \cdots \int_0^{+\infty} P(x) Q(x) \partial_1 \cdots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \prod_{i=1}^n (\gamma_i x_i) \exp\left\{-\sum_{i=1}^n \lambda_i x_i\right\} dx. \end{aligned}$$

Note that

$$\begin{aligned} & \langle x_1^{p_1-1} \cdots x_n^{p_n-1}, x_1^{q_1-1} \cdots x_n^{q_n-1} \rangle_1 \\ &= \int_{\mathbb{R}_+^n} x_1^{p_1+q_1-1} \cdots x_n^{p_n+q_n-1} \partial_1 \cdots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \left(\prod_{i=1}^n \gamma_i \right) \exp\left\{-\sum_{i=1}^n \lambda_i x_i\right\} dx \\ & \quad |p|, |q| \geq 1. \end{aligned}$$

Thus we can construct the other space S_2 which is the space of all linear transformations of the Teugels martingale monomials of the multi-dimensional Gamma process, i.e.

$$\begin{aligned} S_2 = & \left\{ \sum_{p_1+\dots+p_n=d} a_d(p_1, \dots, p_n) \hat{G}(t)^{(p_1, \dots, p_n)} + \sum_{p_1+\dots+p_n=d-1} a_{d-1}(p_1, \dots, p_n) \hat{G}(t)^{(p_1, \dots, p_n)} \right. \\ & \left. + \cdots + \sum_{p_1+\dots+p_n=1} a_1(p_1, \dots, p_n) \hat{G}(t)^{(p_1, \dots, p_n)}, \quad d \geq 1. \right\}. \end{aligned}$$

endowed with the scalar product $\langle \cdot, \cdot \rangle_2$, given by

$$\begin{aligned} & \langle \hat{G}^{(p_1, \dots, p_n)}, \hat{G}^{(q_1, \dots, q_n)} \rangle_2 \\ &= E \left[\left[\hat{G}^{(p_1, \dots, p_n)}, \hat{G}^{(q_1, \dots, q_n)} \right] (1) \right] \\ &= E \left[\hat{G}(1)^{(p_1+q_1, \dots, p_n+q_n)} \right] \\ &= \int_{\mathbb{R}_+^n} x_1^{p_1+q_1-1} \cdots x_n^{p_n+q_n-1} \partial_1 \cdots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \left(\prod_{i=1}^n \gamma_i \right) \exp\left\{-\sum_{i=1}^n \lambda_i x_i\right\} dx \end{aligned}$$

So one clearly sees that $x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} \leftrightarrow \hat{G}^{(p_1, \dots, p_n)}$ is an isometry between S_1 and S_2 . An orthogonalization of $\{x_1^{-1} x_2^{-1} \cdots x_{n-1}^{-1}, x_1^{-1} x_3^{-1} \cdots x_n^{-1}, \dots, x_n^{-1}, 1, x_1, \dots, x_n, x_1^2, x_1 x_2, \dots, x_n^2, \dots\}$ in S_1 can give the multivariate polynomials, so by isometry we also can find an orthogonalization of $\{\hat{G}^{(1,0,\dots,0)}, \dots, \hat{G}^{(0,\dots,0,1)}, \hat{G}^{(2,0,\dots,0)}, \hat{G}^{(1,1,0,\dots,0)}, \dots, \hat{G}^{(0,\dots,0,2)}, \dots\}$.

4.2. The negative multinomial processes

The next process of bounded variation we look at is the negative multinomial processes, sometimes also called Pascal processes. Here the conception of negative multinomial processes can be found in Johnson et al.(1997), and P. Bernardoff (2003).

We define a negative multinomial distribution on \mathbb{N}_0^n . Its distribution is $\sum_{\mathbf{k} \in \mathbb{N}_0^n} prob_k \delta_{\mathbf{k}}$, where $prob_k \delta_{\mathbf{k}}$ denotes the probability measure concentrated at $\mathbf{k} = \{k_1, k_2, \dots, k_n\}$,

$$prob_k \stackrel{\text{def}}{=} \mathbb{P}(k_1, \dots, k_n) = \frac{\Gamma(t + \sum_{i=1}^n k_i)}{k_1! k_2! \dots k_n! \Gamma(t)} \lambda^t \prod_{i=1}^n (\mu \lambda_i)^{k_i},$$

$$k_i = 0, 1, 2, \dots, \quad i = 1, 2, \dots, n.$$

where $0 < \lambda < 1$, $0 < \mu \lambda_i < 1$ for $i = 1, 2, \dots, n$ and $\lambda + \mu(\lambda_1 + \dots + \lambda_n) = 1$.

This type of n-dimensional Lévy processes $P = \{P(t), t \geq 0\}$ where $P(t) = (P_1(t), P_2(t), \dots, P_n(t))$, has a characteristic function given by

$$\begin{aligned} \mathbb{E}[\exp(i\theta' \cdot P(t))] &= \sum_{\mathbf{k} \in \mathbb{N}_0^n} prob_k e^{ik_1 \theta_1} \dots e^{ik_n \theta_n} \\ &= \left(\frac{\lambda}{1 - \mu(\lambda_1 e^{i\theta_1} + \dots + \lambda_n e^{i\theta_n})} \right)^t \end{aligned}$$

The corresponding Lévy measure $\nu(k_1, \dots, k_n)$ is given by

$$\nu(k_1, \dots, k_n) = \frac{(|\mathbf{k}| - 1)!}{k_1! \dots k_n!} \prod_{i=1}^n (\mu \lambda_i)^{k_i}, \quad |\mathbf{k}| \stackrel{\text{def}}{=} k_1 + \dots + k_n.$$

Let us denote with

$$G(t)^{(p_1, \dots, p_n)} \stackrel{\text{def}}{=} \sum_{0 < s \leq t} (\Delta P_1(s))^{p_1} \dots (\Delta P_n(s))^{p_n}, \quad |\mathbf{p}| \geq 1.$$

the power jump processes of P and with $Q^{(p_1, \dots, p_n)} = \{Q^{(p_1, \dots, p_n)}(t), t \geq 0\}$ the corresponding processes of Teugels martingale monomials.

We look for the orthogonalization of the set $\{Q^{(p_1, \dots, p_n)}, |\mathbf{p}| \geq 1\}$ of martingales. The space S_1 is now defined as follows

$$S_1 = \left\{ \sum_{k=1}^d \sum_{|\mathbf{p}|=k} c_k(p_1, \dots, p_n) k_1^{p_1} \dots k_n^{p_n} + c_0 + \sum_{(i_1, \dots, i_n) \in (0, -1]^n, |i| \geq -(n-1)} c_{-1}(i_1, \dots, i_n) k_1^{i_1} \dots k_n^{i_n}; \right.$$

$$\left. d \in \{1, 2, \dots\}, c_j(p_1, \dots, p_n) \in \mathbb{R}, j = -1, 0, \dots, d; k_i \in \mathbb{N}, i = 1, \dots, n; \mathbf{i} = (i_1, \dots, i_n) \right\}$$

endowed with a scalar product $\langle \cdot, \cdot \rangle_1$, given by

$$\langle P(\mathbf{k}), R(\mathbf{k}) \rangle_1 = \sum_{\mathbf{k} \in \mathbb{N}_0^n} P(\mathbf{k}) R(\mathbf{k}) k_1 \dots k_n \frac{(|\mathbf{k}| - 1)!}{k_1! \dots k_n!} \prod_{i=1}^n (\mu \lambda_i)^{k_i}.$$

Note that

$$\begin{aligned} \langle k_1^{p_1-1} \dots k_n^{p_n-1}, k_1^{q_1-1} \dots k_n^{q_n-1} \rangle_1 &= \sum_{\mathbf{k} \in \mathbb{N}_0^n} k_1^{p_1+q_1-1} \dots k_n^{p_n+q_n-1} \frac{(|\mathbf{k}| - 1)!}{k_1! \dots k_n!} \prod_{i=1}^n (\mu \lambda_i)^{k_i} \\ &= p_i, q_j \geq 1, \quad i, j = 1, 2, \dots, n \end{aligned}$$

The other space S_2 is the space of all linear transformations of the Teugels martingales of the negative multinomial processes, i.e.

$$S_2 = \left\{ \sum_{p_1+\dots+p_n=d} a_d(p_1, \dots, p_n) Q(t)^{(p_1, \dots, p_n)} + \sum_{p_1+\dots+p_n=d-1} a_{d-1}(p_1, \dots, p_n) Q(t)^{(p_1, \dots, p_n)} \right. \\ \left. + \dots + \sum_{p_1+\dots+p_n=1} a_1(p_1, \dots, p_n) Q(t)^{(p_1, \dots, p_n)}, \right. \\ \left. d \in \mathbb{N}, \quad (p_1, \dots, p_n) \in \mathbb{N}^n, \quad a_j(p_1, \dots, p_n) \in \mathbb{R}, \quad j = 1, 2, \dots, d \right\}.$$

and is endowed with the scalar product $\langle \cdot, \cdot \rangle_2$, given by

$$\begin{aligned} \langle Q^{(p_1, \dots, p_n)}, Q^{(q_1, \dots, q_n)} \rangle_2 &= \mathbb{E} \left[\left[Q^{(p_1, \dots, p_n)}, Q^{(q_1, \dots, q_n)} \right]_1 \right] \\ &= \mathbb{E} \left[Q^{(p_1+q_1, \dots, p_n+q_n)}(1) \right] \\ &= \sum_{k \in \mathbb{N}^n} k_1^{p_1+q_1} \dots k_n^{p_n+q_n} \frac{(|k| - 1)!}{k_1! \dots k_n!} \prod_{i=1}^n (\mu \lambda_i)^{k_i}. \end{aligned}$$

So one clearly sees that $k_1^{p_1-1} k_2^{p_2-1} \dots k_n^{p_n-1} \leftrightarrow Q^{(p_1, \dots, p_n)}$ is an isometry between S_1 and S_2 . An orthogonalization of $\{k_1^{-1} k_2^{-1} \dots k_{n-1}^{-1}, k_1^{-1} k_3^{-1} \dots k_n^{-1}, \dots, k_n^{-1}, 1, k_1, \dots, k_n, k_1^2, k_1 k_2, \dots, k_n^2, \dots\}$ in S_2 gives the multivariate Meixner polynomials (Griffiths (1975), Griffiths and Spanò (2008), and Koekoek and Swarttouw (1998)), so by isometry we also find an orthogonalization of

$$\{Q^{(1,0,\dots,0)}, \dots, Q^{(0,\dots,0,1)}, Q^{(2,0,\dots,0)}, Q^{(1,1,0,\dots,0)}, \dots, Q^{(0,\dots,0,2)}, \dots\}$$

4.3. The Multivariate Meixner process

A multivariate Meixner process $M(t) = (M_1(t), M_2(t), \dots, M_n(t))^T, t \geq 0$ is a bounded variation Lévy process based on the infinitely divisible distribution. We use the copula to construct a multivariate Meixner process. Here the marginal density functions are given by

$$f_{M_i(t)}(x_i; m_i, a_i) = \frac{(2 \cos(a_i/2))^{2m_i}}{2\pi \Gamma(2m_i)} \exp(a_i x_i) |\Gamma(m_i + i x_i)|^2, \\ x_i \in (-\infty, +\infty), \quad i = 1, 2, \dots, n$$

The corresponding distribution is the measure of orthogonality of the Meixner-Pollaczek polynomials (Koekoek and Swarttouw, 1998). The Meixner process was introduced in Schoutens and Teugels (1998). In Grigelions (1998), it is proposed for a model for risky assets and an analogue of the famous Black and Scholes formula in mathematical finance was established. The marginal characteristic functions of $M_i(t)$, $i = 1, 2, \dots, n$, are given by

$$E[\exp(\theta_i M_i(t))] = \left(\frac{\cos(a_i/2)}{\cosh((\theta_i - i a_i)/2)} \right)^{2m_i t}, \quad i = 1, 2, \dots, n.$$

In according to the results in Schoutens and Teugels (1998) and Schoutens (1999) and applying (11), its Lévy measure can be calculated as:

$$\begin{aligned} \nu(dx) &= \partial_1 \dots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \prod_{i=1}^n \frac{m_i \exp(a_i x_i)}{x_i \sinh(\pi x_i)} dx_i \\ &= \partial_1 \dots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \prod_{i=1}^n m_i |\Gamma(1 + i x_i)|^2 \frac{\exp(a_i x_i)}{\pi x_i^2} dx_i. \end{aligned}$$

Also note that

$$\left(\prod_{i=1}^n x_i^2 \right) \nu(dx) = \partial_1 \cdots \partial_n F|_{\xi_1=U_1(x_1), \dots, \xi_n=U_n(x_n)} \prod_{i=1}^n m_i |\Gamma(1 + ix_i)|^2 \frac{\exp(a_i x_i)}{\pi} dx_i.$$

Being completely similar as in the above two examples, we can orthogonalize the multivariate Teugels martingales for the multivariate Meixner process by isometry.

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